Question Paper Code : 5637

M.A. / M.Sc. (Semester - II) Examination, 2018

MATHEMATICS

[Second Paper]

(Module Theory)

Time : Three Hours] **[Maximum Marks:70**

Note : Answer **five** questions in all. Question **No.1** is **compulsory**. Besides this, attempt **one** question from each Unit.

1. Attempt all parts : [3x10=30]

   (a) If X is a submodule of M, then show that Ann(X) is an ideal of R

   (b) Define a free module and show that Z- module Q is not free.

   (c) Define divisible group with an example

   (d) If L is a submodule of M, then show that $0 \to L \to M \to M/L \to 0$ is a short exact sequence.

(e) Let $f : R \rightarrow S$ be a ring homomorphism. If M is S-module then show that M is also an R - Module.

(f) If G is a finite abelian group of order $n \geq 2$, then show that G is not a free Z-module.

(g) Show that every homomorphic image of a divisible group is also a divisible group.

(h) If M be an R-module then prove that there exist a free R-module F and a submodule K of F such that $F/K \approx M$

(i) Define a split exact sequence. Give an example of a short exact sequence which is not split exact.

(j) Define an injective module and give an example.

UNIT-I

2. (a) If M be a non-empty and simple R-module then show that $End_R(M)$ is a division ring. [5]

(b) If $f: M \rightarrow N$ be an R-module homomorphism, then show that : $M/Ker\, f \approx Im f$. [5]

3. (a) Prove that a non-zero simple R-module is always cyclic. Is the converse true ? Justify your answer with some suitable example. [5]

(b) Let M_1, M_2, M_3 ..... Mn be an R-modules. Prove that M is the direct sum of M_1, M_2, M_3 .... Mn if and only if for each I in { 1,2,3,....n}, there exist an R-module hompmorphism sp1 : M $\to M_l$ and $u_l : M_l \to$ M such that [5]

(i) $p_l u_l = 1_{M_l}$

(ii) $p_k u_l = 0$ for k $\neq 1$

(ii) $\sum u_l\, p_l = 1_M$

UNIT-II

4. (a) Let $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ be short exact sequence of R-modules. Prove that there exist an R-mnodule homomorphism k: $M_2 \to$ M such that gk = I_{M2}. If and only if M is the direct sum of M_1 and M_2 [5]

(b) Show that every module over a division ring is always free. [5]

5. (a) If M be a free R-module with basis B and N be an R-module and f:B$\to$N is a mapping then show that there exist a unique R-module homomorphism μ M$\to$N such that $^{\mu}/_{B}$ = f. [5]

(b) If F be a free R-module then prove that every short exact sequence of R-modules $0 \to N \to M \to F \to 0$ is split exact sequence. [5]

UNIT-III

6. (a) If M be a module over a division ring D then prove that any two bases of M have the same cardinality. [5]

(b) Prove that every finitely generated torsion free module over a P.I.D. is free. [5]

7. (a) If M is finitely generated torsion free module over a P.I.D. then show that :

$M \approx T(M) + M/T(M)$ [5]

(b) Let R be a P.I.D. and M be a free module with basis $B = \{xi / i \in I\}$; then show that : [5]

(i) $x = \sum r_i x_i \in M \setminus \{0\}$ is primitive if and only if gcd $(r_i \setminus i \in I) = 1$

(ii) If $y = \sum s_i x_i \in M \setminus \{0\}$ and d = gcd $(s_i \setminus i \in I)$; then y = dy' and y' is primitive element of M.

UNIT-IV

8. (a) Define a projective module and show that every free module is projective module. Is the converse true ? [5]

(b) Let Pi be a family of R-modules and let $P = \prod P_i$ show that : P is projective module if and only if each P_i is projective module; for all i. [5]

9. (a) Show that the following statements are equivalent [5]

(i) P is Projective R-module

(ii) if $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} P \to 0$ be a short exact sequence, then it splits.

(iii) There exist an R-module K such that $\overline{P \oplus K}$ is free. (Where $\overline{P \oplus K}$ stands for direct sum of P and K)

(b) Prove that an abelian group is divisible if and only if it is an injective Z-module. [5]

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